INTRODUCTION TO PLATE BENDING THEORY

BY

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Abstract: This paper presents an overview of the governing equations for the bending study of the types of plates, with several known plate theories from the literature. For the high order theories (Mindlin and Reissner), which considers, shear deformations through the thickness of a plate, is theoretically compared to the classical plate theory (Kirchhoff).

Keywords: Stress, Strain, Deflection, Normal to Surface, Shear Stress, Transverse Strain, Twist, Moment

1. Introduction

A plate is a planer structure with a very small thickness in comparison to the planer dimensions. The forces applied on a plate are perpendicular to the plane of the plate. Therefore, plate resists the applied load by means of bending in two directions and twisting moment. A plate theory takes advantage of this disparity in length scale to reduce the full three-dimensional solid mechanics problem to a two-dimensional problem. The aim of plate theory is to calculate the deformation and stresses in a plate subjected to loads. A flat plate, like a straight beam carries lateral load by bending. The analyses of plates are categorized into two types based on thickness to breadth ratio: thick plate and thin plate analysis. If the thickness to width ratio of the plate is less than 1/20 and the maximum deflection is less than one tenth of thickness, then the plate is classified as thin plate. The well known as Kirchhoff plate theory is used for the analysis of such thin plates. On the other hand, Mindlin–Reissner plate theory is used for thick plate where the effect of shear deformation is included.

2. Notations and Sign Conventions

Let consider plates to be placed in XY plane. Representation of plate surface slopes $W_{,x}$, $W_{,y}$ by right hand rule produces arrows that point in negative Y and positive X directions respectively. Both surface slopes and rotations are required for plate elements. Signs and subscripts of rotations and slopes are reconciled by replacing $\boldsymbol{\theta_x}$ by $\boldsymbol{\Psi_y}$ and $\boldsymbol{\theta_y}$ by $\boldsymbol{-\Psi_x}$

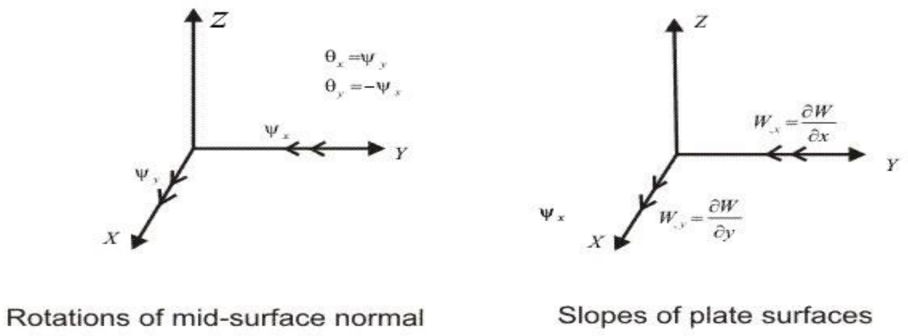

Figure 1:Notations and sign conventions

3. Thin Plate Theory

The Kirchhoff–Love theory of plates or Classical thin plate theory is a two-dimensional mathematical model that is used to determine the stresses and deformations in thin plates subjected to forces and moments. This theory is an extension of Euler-Bernoulli beam theory and was developed in 1888 by Love using assumptions proposed by Kirchhoff. The theory assumes that a mid-surface plane can be used to represent a three-dimensional plate in two-dimensional form.

The following kinematic assumptions that are made in this theory:

- straight lines normal to the mid-surface remain straight after deformation
- straight lines normal to the mid-surface remain normal to the mid-surface after deformation
- the thickness of the plate does not change during a deformation.

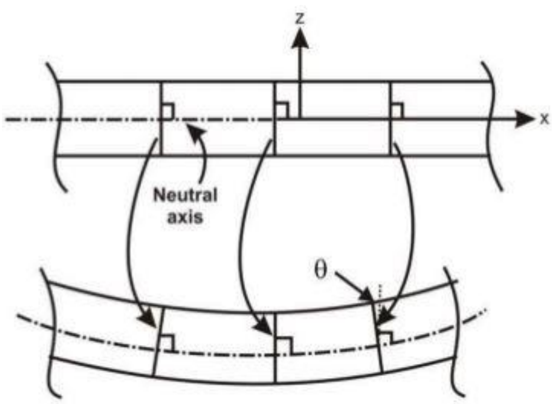

Figure 2:Kirchhoff plate after bending

3.1. Basic relationships

Let, a plate of thickness t has mid-surface at a distance $t/2$ from each lateral surface. For the analysis purpose, X-Y plane is located in the plate mid-surface, therefore z=0 identifies the mid-surface. Let **u, v,w** be the displacements at any point (x, y, z).

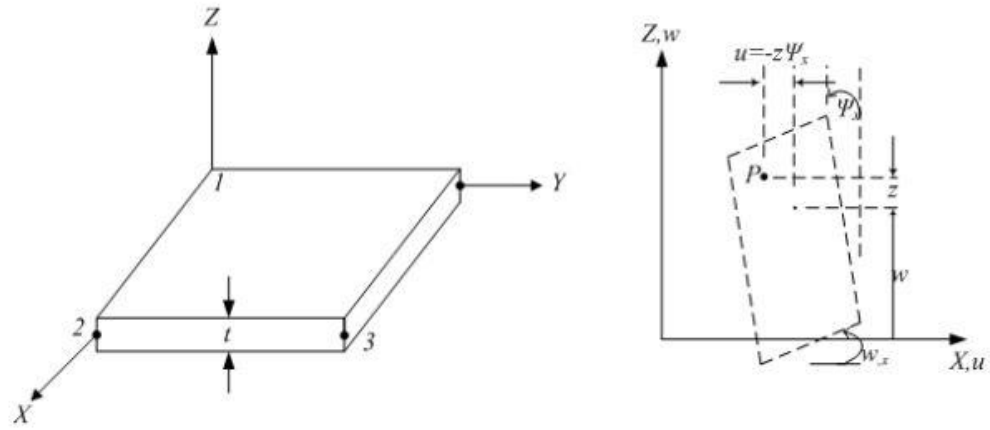

Figure 3: Thin Plate Element

Then the variation of u and v across the thickness can be expressed in terms of displacement w as

$$u = -z\frac{\partial w}{\partial x} \qquad\qquad v = -z\frac{\partial w}{\partial y} \tag{1}$$

Where, w is the deflection of the middle plane of the plate in the z direction. Further the relationship between, the strain and deflection are given by,

$$\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} = z\,\chi_x \\
\varepsilon_y &= \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2} = z\,\chi_y \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y} = z\,\chi_{xy}
\end{aligned} \tag{2}$$

where,

ε corresponds to direct strain

γ corresponds to shear strain

χ corresponds to curvature along respective directions.

Or in matrix form, the above expression can be written as

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = -z\begin{bmatrix} \frac{\partial^2}{\partial x^2} \\ \frac{\partial^2}{\partial y^2} \\ \frac{\partial^2}{\partial x \partial y} \end{bmatrix} w \tag{3}$$

Or, $\varepsilon = -z\Delta w$ (4)

Where, ε is the vector of in-plane strains, and Δ is the differential operator matrix.

3.2. Constitutive equations

From Hooke's Law,

$$\sigma=[D]\varepsilon \tag{5}$$

Where,

$$[D]=\frac{E}{\left(1-\upsilon^2\right)}\begin{bmatrix}1 & \upsilon & 0\\ \upsilon & 1 & 0\\ 0 & 0 & \dfrac{1-\upsilon}{2}\end{bmatrix} \tag{6}$$

Here, [D] is equal to the value defined for 2D solids in plane stress condition (i.e., σ_z = 0).

3.3. Calculation of moments and shear forces

Let consider a plate element of $dx \times dy$ and with thickness t. The plate is subjected to external uniformly distributed load p. For a thin plate, body force of the plate can be converted to an equivalent load and therefore, consideration of separate body force is not necessary. By putting eq. (4) in equation (5),

$$\sigma=-z\,[D]_{\Delta}w \tag{7}$$

It is observed from the above relation that the normal stresses are varying linearly along thickness of the plate (Fig 4). Hence the moments (Fig 5) on the cross section can be calculated by integration.

$$M=\begin{Bmatrix}M_x\\ M_y\\ M_{xy}\end{Bmatrix}=\int_{-t/2}^{t/2}\sigma z dt=-\left(\int_{-t/2}^{t/2} z^2 dt\right)[D]\Delta w=-\frac{t^3}{12}[D]\Delta w \tag{8}$$

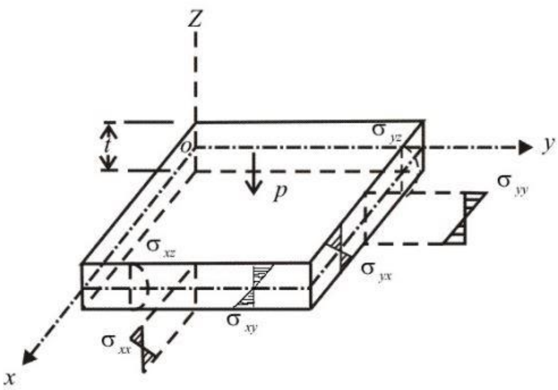

Figure 4: Stresses in Thin Plate

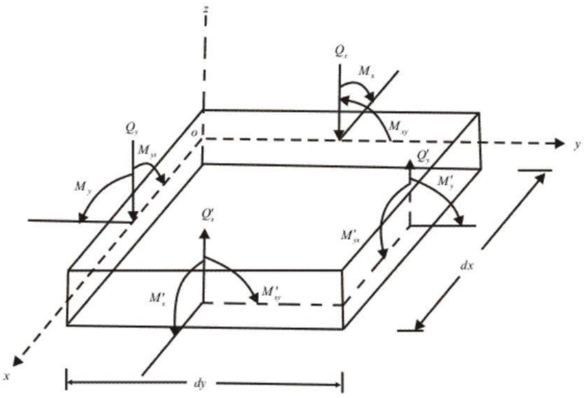

Figure 5: Forces and Moment in Thin Plate

On expansion of eq. (8) one can find the following expressions.

$$M_x = -\frac{Et^3}{12\left(1-\upsilon^2\right)}\left(\frac{\partial^2 w}{\partial x^2}+\upsilon\frac{\partial^2 w}{\partial y^2}\right) = D_P\left(\chi_x+\upsilon\chi_y\right)$$

$$M_y = -\frac{Et^3}{12\left(1-\upsilon^2\right)}\left(\frac{\partial^2 w}{\partial y^2}+\upsilon\frac{\partial^2 w}{\partial x^2}\right) = D_P\left(\chi_y+\upsilon\chi_x\right)$$

$$M_{xy} = M_{yx} = \frac{Et^3}{12\left(1+\upsilon\right)}\frac{\partial^2 w}{\partial x\partial y} = -\frac{D_P\left(1-\upsilon\right)}{2}\chi_{xy} \tag{9}$$

Where, D_P is known as flexural rigidity of the plate and is given by,

$$D_P = \frac{Et^3}{12\left(1-\upsilon^2\right)} \tag{10}$$

Let consider the bending moments vary along the length and breadth of the plate as a function of *x* and *y*. Thus, if M_x acts on one side of the element, M_x' acts on the opposite side. Considering equilibrium of the plate element, the equations for forces can be obtained as

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+p=0 \tag{11}$$

$$\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}=Q_x \tag{12}$$

$$\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}=Q_y \tag{13}$$

Using eq. 9 in eqs 12 & 13, the following relations will be obtained.

$$Q_x = -D_P\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right) \tag{14}$$

$$Q_y = -D_P\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right) \tag{15}$$

Using eqs. (14) and (15) in eq. (11) following relations will be obtained.

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}=-\frac{p}{D_P} \tag{16}$$

4. Thick Plate Theory

The Mindlin–Reissner theory of plates is an extension of Kirchhoff–Love plate theory that considers shear deformations through-the-thickness of a plate. The theory was proposed in 1951 by Raymond Mindlin. A similar, but not identical, theory had been proposed earlier by Eric Reissner in 1945. Both theories are intended for thick plates in which the normal to the mid-surface remains straight

but not necessarily perpendicular to the mid-surface. Both theories include in-plane shear strains and both are extensions of Kirchhoff–Love plate theory incorporating first-order shear effects.

Mindlin's theory assumes that there is a linear variation of displacement across the plate thickness but that the plate thickness does not change during deformation. An additional assumption is that the normal stress through the thickness is ignored; an assumption which is also called the plane stress condition. On the other hand, Reissner's theory assumes that the bending stress is linear while the shear stress is quadratic through the thickness of the plate. This leads to a situation where the displacement through-the-thickness is not necessarily linear and where the plate thickness may change during deformation. Therefore, Reissner's theory does not invoke the plane stress condition. It basically depends on following assumptions,

- The deflections of the plate are small
- Stresses normal to the mid-surface are negligible.
- Normal to the plate mid-surface before deformation remains straight but is not necessarily normal to it after deformation.

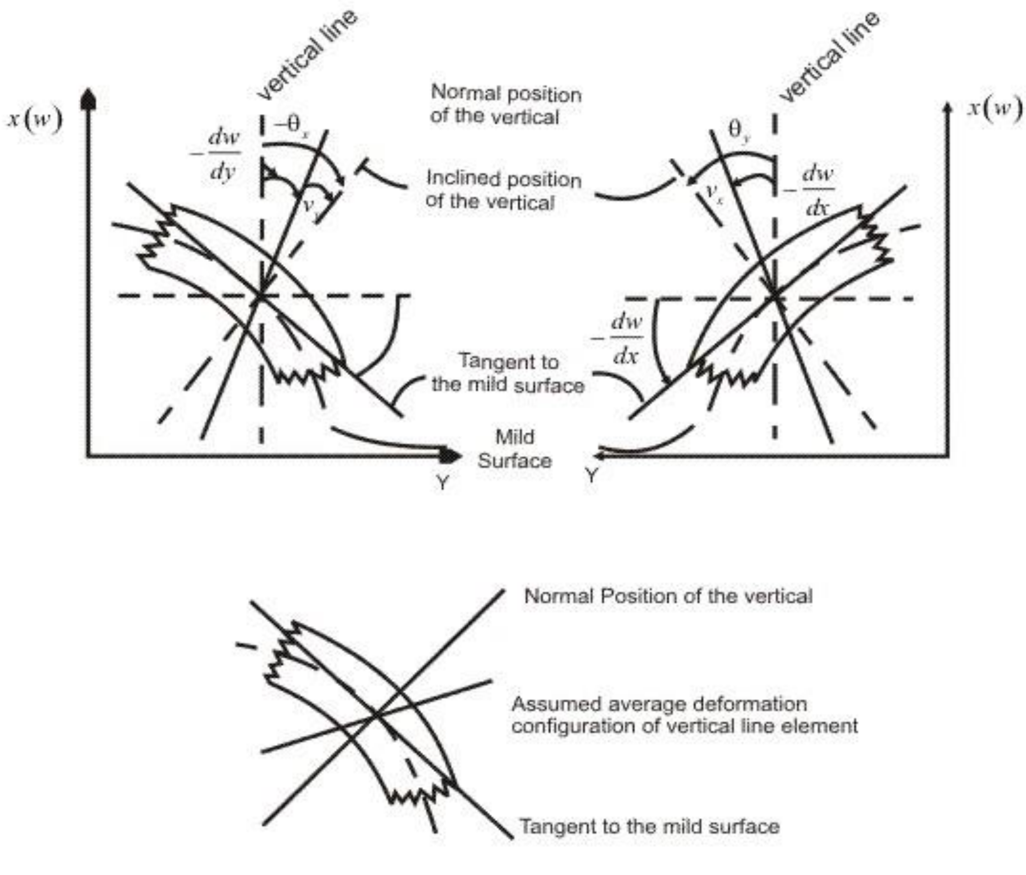

Figure 6:Bending of Thick Plate

The Mindlin–Reissner theory is often called the first-order shear deformation theory of plates. Since a first-order shear deformation theory implies a linear displacement variation through the thickness, it is incompatible with Reissner's plate theory.

Thus, according to Mindlin plate theory, the deformation parallel to the undeformed mid surface, u and v, at a distance z from the centroidal axis are expressed by,

$$u = z\theta y \tag{17}$$

$$v = -z\theta x \tag{18}$$

Where θ_x and θ_y are the rotations of the line normal to the neutral axis of the plate with respect to the *x* and *y* axes respectively before deformation.

The curvatures are expressed by

$$\chi_x = \frac{\partial \theta_y}{\partial x} \tag{19}$$

$$\chi_y = -\frac{\partial \theta_x}{\partial y} \tag{20}$$

Similarly, the twist for the plate is given by,

$$\chi_{xy} = \left(\frac{\partial \theta_y}{\partial y} - \frac{\partial \theta_x}{\partial x} \right) \tag{21}$$

Using eqs 9 and 10, the bending stresses for the plate is given by

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \frac{Et^3}{12(1-\upsilon^2)} \begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \frac{1-\upsilon}{2} \end{bmatrix} \begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix} \tag{22}$$

Or, $\{M\}=[D]\{\chi\}$ (23)

Further, the transverse shear strains are determined as

$$\gamma_{xz} = \theta_y + \frac{\partial w}{\partial x} \tag{24}$$

$$\gamma_{yz} = -\theta_x + \frac{\partial w}{\partial y} \tag{25}$$

The shear strain energy can be expressed as

$$\begin{aligned} U_s &= \frac{1}{2}\alpha GA \iint_A \left[(\gamma_x)^2 + (\gamma_y)^2 \right] dx\, dy \\ &= \frac{1}{2}\alpha GA \iint_A \left[\left(\theta_y + \frac{\partial w}{\partial x} \right)^2 + \left(-\theta_x + \frac{\partial w}{\partial y} \right)^2 \right] dx\, dy \end{aligned} \tag{26}$$

Where, $G = \frac{E}{1+\mu}$The shear stresses are

$$\begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \frac{E}{2(1+\mu)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \gamma_x \\ \gamma_y \end{Bmatrix} \tag{27}$$

Hence the resultant shear stress is given by,

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \frac{Et\alpha}{2(1+\mu)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \gamma_x \\ \gamma_y \end{Bmatrix} \tag{28}$$

Or,

$$\{Q\} = [D_s]\{\gamma\} \tag{29}$$

Here "α" is the numerical correction factor used to characterize the restraint of cross section against warping. If there is no warping i.e., the section is having complete restraint against warping then α = 1 and if it is having no restraint against warping then α = 2/3. The value of α is usually taken to be $\pi^2/12$ or 5/6. Now, the stress resultant can be combined as follows.

$$\begin{Bmatrix} \{M\} \\ \{Q\} \end{Bmatrix} = \begin{bmatrix} [D] & 0 \\ 0 & [D_s] \end{bmatrix} \begin{Bmatrix} \{\chi\} \\ \{\gamma\} \end{Bmatrix} \tag{30}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \\ Q_x \\ Q_y \end{Bmatrix} = \begin{bmatrix} \frac{Et^3}{12(1-\mu^2)} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \frac{Et}{2(1+\mu)} \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} \end{bmatrix} \begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \\ \gamma_x \\ \gamma_y \end{Bmatrix} \tag{31}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \\ Q_x \\ Q_y \end{Bmatrix} = \begin{bmatrix} \frac{Et^3}{12(1-\mu^2)} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \frac{Et}{2(1+\mu)} \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} \end{bmatrix} \begin{Bmatrix} \frac{\partial \theta_y}{\partial x} \\ -\frac{\partial \theta_x}{\partial y} \\ \frac{\partial \theta_y}{\partial y} - \frac{\partial \theta_x}{\partial x} \\ \theta_y + \frac{\partial w}{\partial x} \\ -\theta_x + \frac{\partial w}{\partial y} \end{Bmatrix} \tag{32}$$

The above relation may be compared with usual stress-strain relation. Thus, the stress resultants and their corresponding curvature and shear deformations may be considered analogous to stresses and strains.

5. Boundary Conditions

For different boundaries of the plate (Fig. 7), suitable conditions are to be incorporated in plate equation for solving the governing differential equations. For example, following conditions need to satisfy along y direction of the plate for various boundaries.

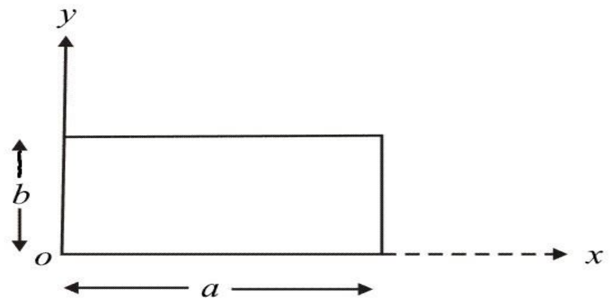

Figure 7: Plate with four boundaries

- Simply support edge (Along *y* direction)
  $w(x,y) = 0$, $M_x = 0$ [$x = const$ & $0 \leq y \leq b$]

- Clamped Edge (Along *y* direction)
  $W(x,y)=0, \frac{dw}{dx}(x,y)=0$ [$x = const$ & $0 \leq y \leq b$]

- Free Edge (Along y direction)
  $M_x=0$, $Q_x+ \frac{dMxy}{dx} = 0$ [$x = const$ & $0 \leq y \leq b$]

Similar to the above, the boundary conditions along x direction can also be obtained. Once the displacements w(x,y) of the plate at various positions are found, the strains, stresses and moments developed in the plate can be determined by using corresponding equations.

6. Applications and Examples

For a variety of boundary conditions, the displacement of plates along the thickness direction, t, may be taken as different expressions to satisfy the given boundary conditions and then be determined using the differential equations as mentioned in above equations.

For example, for a rectangular thin plate with four simply supported sides, the Navier's solution may be adopted; for such a plate with two opposite simply supported sides, the Levy solution may be adopted. However, in some cases, it is convenient to use the displacement variation method to solve such a problem. For example, for a rectangular thin plate with four sides fixed, the Galerkin approach may be adopted; for such a plate with two opposite sides fixed, the Ritz approach may be adopted.

7. Conclusion

The Mindlin plate theory (or thick plate theory or shear deformation theory) allow for possible transverse shear strains. In this theory, there is the added complication that vertical line elements before deformation do not have to remain perpendicular to the mid-surface after deformation, although they do remain straight. Thus, shear strains ε_{yz} and ε_{zx} are generated, constant through the thickness of the plate. The classical plate theory is inconsistent in the sense that elements are assumed to remain perpendicular to the mid-plane, yet equilibrium requires that stress components σ_{xz} , σ_{yz} still arise (which would cause these elements to deform). The theory of thick plates is more consistent, but it still assumes that $\sigma_{zz} = 0$. Note that both are approximations of the exact three-dimensional equations of elasticity.

References

- https://en.wikipedia.org/wiki/Bending_of_plates
- www.nptel.ac.in/courses/105105041/module%206
- https://web.stanford.edu/.../Introduction%20to%20the%20Theory%20of%20Plates
- Part II: Engineering Solid Mechanics by Piaras Kelly; Publisher: The University of Auckland 2013